Technical Notes

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A Variation on MacCormack's Method for Axisymmetric Viscous Compressible Flows

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Nomenclature

E	= vector of convective fluxes
$\mathcal{E}$	= total specific internal energy
F	= radial convective flux
H	= vector of inhomogeneous terms
k	= thermal conductivity
P	= pressure
t	= time
Δt	= discrete time interval
T	= temperature
u_1, u_2	= axial and radial velocity components
U	= vector of conserved quantities
x_1, x_2 or x, r	= axial and radial coordinates
Δx or Δr	= discrete spatial increments
δ_{kl} or $\delta^{\alpha\beta}$	= Kronecker delta
ϵ	= radial truncation error
μ, λ	= viscosity coefficients
π	= modified pressure
ρ	= density
τ	= stress tensor
Δ, ∇	= forward and backward finite difference operators

Superscripts

N	= discretized time index; $t=(N-1)\Delta t$
T	= transpose of a vector
α	= equation index = 1,...,4
$\bar{(\ )}$	= provisional values

Subscripts

i, j	= spatial coordinate index
I, J	= spatial discrete index; $x=(I-1)\Delta x$, $r=(J-1)\Delta r$
x, r	= difference operator indicator

Introduction

RECENT advances in computational fluid dynamics and computer technology have made possible the solution of complicated three-dimensional flowfields. Still, if the symmetry of the problem permits, the use of the axisymmetric approximation provides considerable computational economy and greater resolution. However, if the flowfield in the vicinity of the axis is to be calculated, great care must be taken in choosing a finite difference scheme which properly models the axisymmetric form of the governing equations. The purpose of this Note is to present such a finite difference scheme which maintains all the desirable characteristics of MacCormack's highly successful explicit technique.[1]

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Numerical Technique

The finite difference technique to be presented is used to solve flowfield problems governed by the axisymmetric approximation to the covariant form of the conservation equations of mass, momentum, and energy,

$$\frac{\partial U^\alpha}{\partial t}+\frac{\partial E_1^\alpha}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rE_2^\alpha)+H^\alpha=0 \tag{1}$$

where

$$U=(\rho,\ \rho u_1, \rho u_2, \rho\mathcal{E})^T$$

$$E_i^\alpha=u_iU^\alpha-\tau_{i1}\delta^{\alpha 2}-\tau_{i2}\delta^{\alpha 3}-(u_j\tau_{ij}+q_i)\delta^{\alpha 4}$$

$$H^\alpha=[\partial\pi/\partial r+2\mu u_2/r^2]\delta^{\alpha 3}$$

$$\tau_{ij}=\mu(\partial u_i/\partial x_j+\partial u_j/\partial x_i)-\pi\delta_{i1}\delta_{j1}$$

$$\pi=P-\lambda(\partial u_i/\partial x_i+u_2/r)$$

$$q_i=k\partial T/\partial x_i-u_2\pi\delta_{i2}$$

and summation over repeated indices is implied. Equations (1) will be referred to as the Euler equations for $\mu=\lambda=k=0$ and the Navier-Stokes equations otherwise.

The desired finite difference equation should preserve the conservation law form of the differential equation throughout the flowfield in order to resolve regions of high flow gradients. It should also be stable and of second order accuracy in space and time.

MacCormack's explicit technique exhibits these properties and has been quite successful in computing flowfields in both two- and three-dimensional Cartesian frames. Thus, it becomes a likely candidate for discretizing Eqs. (1). The straightforward application of this technique on a uniform mesh yields

$$\bar{U}_{IJ}=U_{IJ}^N-\Delta t\left[\Delta_x E_{IJ}^N+\frac{1}{r_J}\Delta_r(rF)_{IJ}^N+H_{IJ}^N\right]$$

$$U_{IJ}^{N+1}=\tfrac{1}{2}\left[U_{IJ}^N+\bar{U}_{IJ}-\Delta t\left(\nabla_x\bar{E}_{IJ}+\frac{1}{r_J}\nabla_r(\overline{rF})_{IJ}+\bar{H}_{IJ}\right)\right] \tag{2}$$

where for convenience the equation index has been dropped and E and F have been substituted for E_1^α and E_2^α respectively. The diffusive terms in E and F are evaluated using the opposing difference operator and the pressure derivative in H is evaluated with the operator appropriate to the step.

The differencing in the axial direction satisfies all our criteria since it is identical to the Cartesian scheme. H is also of second order accuracy in r. However, let us examine the discrete model of the radial term. Expressing the difference operators in explicit notation yields

$$\frac{1}{r_J}\Delta_r(rF)_J=\frac{1}{r_J\Delta r}(r_{J+1}F_{J+1}-r_JF_J)$$

$$\frac{1}{r_J}\nabla_r(rF)_J=\frac{1}{r_J\Delta r}(r_JF_J-r_{J-1}F_{J-1}) \tag{3}$$

where the inactive indices have been suppressed. Substituting Eqs. (3) into Eqs. (2) and expanding in a Taylor series shows that the lowest order radial truncation error is

$$\epsilon = \frac{1}{2}\frac{\Delta r^2}{r_J}\left(\frac{\partial^2 F}{\partial r^2}\right)_J$$

This brings to light a pernicious quality of the direct application of MacCormack's technique; namely, that the error grows in inverse proportion to r. That is to say that the truncation error becomes a first order diffusion-like error at $r = \Delta r$ (i.e., one node above the axis). Further, this error exists regardless of the boundary conditions used at the axis. Clearly, applying the radial differences of Eqs. (3) is inappropriate near the axis.

MacCormack[1] has suggested a modification of his method for use in axisymmetric problems. In it r is evaluated at a different point than F and the r^{-1} term is always lagged by one-half a grid spacing. Unfortunately, this method is only valid for $r \gg \Delta r$ and produces similar errors near the axis.

The method presented here takes a similar approach; but, as will be seen, produces a finite difference analog valid throughout the field. This method replaces the difference operators of Eqs. (3) with the following

$$\frac{1}{r_J}\Delta_r(rF)_J = \frac{1}{r_J\Delta r}(r_J F_{J+1} - r_{J-1}F_J)$$

$$\frac{1}{r_J}\nabla_r(rF)_J = \frac{1}{r_J\Delta r}(r_{J+1}F_J - r_J F_{J-1}) \qquad (4)$$

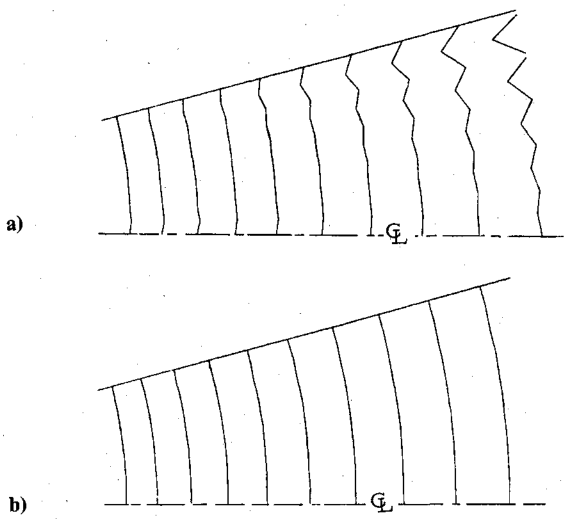

Fig. 1 Mach number contours for inviscid axisymmetric source flow computed using a) Eqs. (3) and b) Eqs. (4).

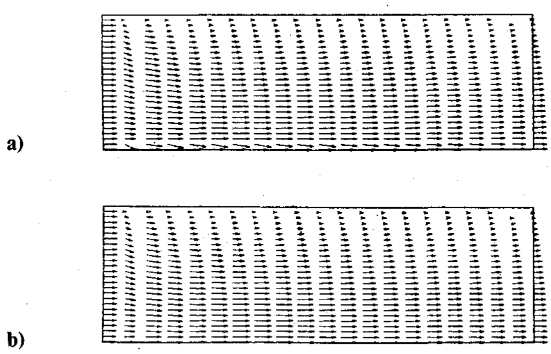

Fig. 2 Velocity vectors for viscous axisymmetric pipe flow computed using a) Eqs. (3) and b) Eqs. (4).

Note here that r *lags* F by one grid spacing in the forward operation and it *leads* F in the backward operation.

The truncation error resulting from the use of Eqs. (4) is

$$\epsilon = \frac{1}{6}\Delta r^2\left(\frac{\partial^3 F}{\partial r^3}\right)_J$$

This is identical to the error generated in the Cartesian case and is independent of r.

Von Neumann stability analysis shows all the methods discussed to be stable with a typical Courant-Friedrichs-Lewy (CFL) stability limit. Thus, the use of Eqs. (3) yields a stable method which produces erroneous results. The new technique is both stable and accurate.

Results

The method outlined above has been successfully incorporated in computer codes which solve the Navier-Stokes and Euler equations by both relaxation[2] and spatial marching[3] techniques. It has also been used in a rocket engine nozzle/plume code.[4]

The results of two example cases are presented here. The first is the solution of a simple inviscid axisymmetric source flow problem using the Euler equations. It consists of a 15-deg expansion of Mach 2 flow in an axisymmetric duct. Figure 1 shows the Mach number contours calculated using both Eqs. (3) and (4). Both results were run at a Courant number of 1 and converged to machine accuracy. The method presented here yields the expected circular Mach contours. The use of Eqs. (3) yields highly inaccurate results. The errors in the neighborhood of the axis have convected throughout the flowfield.

The second example is a subsonic boundary-layer growth in a circular pipe. Figure 2 shows the velocity vectors using both schemes. It readily can be seen that the method presented here corrects the inaccuracies in the vicinity of the axis.

Conclusion

A finite difference analog of the axisymmetric approximation to the Navier-Stokes equations has been presented. The method, based on MacCormack's explicit technique, maintains second order accuracy in all terms and has been successfully applied to both viscous and inviscid flow calculations.

Acknowledgments

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References

[1] MacCormack, R.W., "Effect of Viscosity in Hypervelocity Impact Cratering," AIAA Paper 69-354, Cincinnati, Ohio, 1969.

[2] Spradley, L.W., Stalnaker, J.F., and Ratliff, A.W., "Solution of the Three-Dimensional Navier-Stokes Equations on a Vector Processor," *AIAA Journal*, Vol. 19, Oct. 1981, pp. 1302-1308.

[3] Spradley, L. W., Stalnaker, J. F., and Xiques, K. E., "Quasi-Parabolic Technique for Spatial Marching Navier-Stokes Computations," *AIAA Journal*, Vol. 20, Jan. 1982, pp. 31-32.

[4] Smith, S.D., "Improved Rocket Engine Nozzle/Plume Code," Paper presented at the JANNAF 13th Plume Technology Meeting, Houston, Texas, April 1982.